The monotonicity condition for Backward Stochastic Differential Equations on Manifolds

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Abstract

In [1] and [2], we studied the problem of the existence and uniqueness of a solution to some general BSDE on manifolds. In these two articles, we assumed some Lipschitz conditions on the drift $f(b,x,z)$. The purpose of this article is to extend the existence and uniqueness results under weaker assumptions, in particular a monotonicity condition in the variable x. This extends well-known results for Euclidean BSDE.

1 Reminder of the problem

Unless otherwise stated, we shall work on a fixed finite time interval $[0;T]$; moreover, $(W_t)_{0\leq t\leq T}$ will always denote a Brownian Motion (BM for short) in $\mathbb{R}^{d_w}$, for a positive integer d_w. Moreover, Einstein's summation convention will be used for repeated indices in lower and upper position.

Let $(B_t^y)_{0\leq t\leq T}$ denote the $\mathbb{R}^d$-valued diffusion which is the unique strong solution of the following SDE :

$$\left\{\begin{array}{rcl} dB_t^y &=& b(B_t^y)dt+\sigma(B_t^y)dW_t \\ B_0^y &=& y, \end{array}\right. \tag{1.1}$$

where $\sigma\ :\mathbb{R}^d\to\mathbb{R}^{d\times d_w}$ and $b:\mathbb{R}^d\to\mathbb{R}^d$ are C^3 bounded functions with bounded partial derivatives of order 1, 2 and 3.

Let us recall the problem studied in [1] and [2]. We consider a manifold M endowed with a connection Γ, which defines an exponential mapping. On M, we study the uniqueness and existence of a solution to the equation (under infinitesimal form)

$$(M+D)_0 \left\{ \begin{array}{l} X_{t+dt} = \exp_{X_t}(Z_t dW_t + f(B_t^y, X_t, Z_t)dt) \\ X_T = U \end{array} \right.$$

where $Z_t \in \mathcal{L}(\mathbb{R}^{d_w}, T_{X_t}M)$ and $f(B_t^y, X_t, Z_t) \in T_{X_t}M$.

For details about links with PDEs, the reader is referred to the introductions of [1] and [2].

In local coordinates (x^i), the equation $(M+D)_0$ becomes the following backward stochastic differential equation (BSDE in short)

$$(M+D) \left\{ \begin{array}{l} dX_t = Z_t dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k|[Z_t]^j) + f(B_t^y, X_t, Z_t)\right) dt \\ X_T = U. \end{array} \right.$$

We keep the same notations as in [1] : $(\cdot|\cdot)$ is the usual inner product in an Euclidean space, the summation convention is used, and $[A]^i$ denotes the i^{th} row of any matrix A; moreover,

$$\Gamma_{jk}(x) = \begin{pmatrix} \Gamma^1_{jk}(x) \\ \vdots \\ \Gamma^n_{jk}(x) \end{pmatrix} \tag{1.2}$$

is a vector in $\mathbb{R}^n$, whose components are the Christoffel symbols of the connection. We keep the notations Z_t for a matrix in $\mathbb{R}^{n\times d_w}$ and f for a mapping from $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d_w}$ to $\mathbb{R}^n$. The process X will take its values in a compact set, and a solution of equation $(M+D)_0$ will be a pair of processes (X, Z) in $M \times (\mathbb{R}^{d_w} \otimes TM)$ such that X is continuous and $\mathbb{E}(\int_0^T \|Z_t\|_r^2 dt) < \infty$ for a Riemannian norm $\|\cdot\|_r$; in global coordinates $O \subset \mathbb{R}^n$, $(X,Z) \in O \times \mathbb{R}^{n\times d_w}$ and $\mathbb{E}(\int_0^T \|Z_t\|^2 dt) < \infty$ (see below for the definitions of the norms).

We gave in [1] and [2] existence and uniqueness results for the solutions of the BSDE $(M+D)$ for a drift verifying some geometrical Lipschitz condition in the variables (b, x, z). Here we propose to weaken these assumptions, replacing the Lipschitz condition in the variable x by a monotonicity condition and some other mild conditions; see Section 3 below. The results for this condition were announced in [2].

In Section 2, we recall the general hypothesis. We give the monotonicity condition and other assumptions on the drift f in Section 3, as well as existence and uniqueness theorems under the current assumptions. In Section 4, we sketch the changes to make in the uniqueness proofs in [1] and [2], to get uniqueness in the new context, and in Section 5, we prove the existence results. At the end in Section **??**, we give the corresponding results for random terminal times.

2 Notations and hypothesis

In all the article, we suppose that a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le T})$ (verifying the usual conditions) is given (with $T < \infty$ a deterministic time) on which $(W_t)_t$ denotes a d_w-dimensional BM. Moreover, we always deal with a complete Riemannian manifold M of dimension n, endowed with a linear symmetric (i.e. torsion-free) connection whose Christoffel symbols Γ^i_{jk} are smooth; the connection does not depend *a priori* on the Riemannian structure.

On M, δ denotes the Riemannian distance; $|u|_r$ is the Riemannian norm for a tangent vector u and $|u'|$ the Euclidean norm for a vector u' in $\mathbb{R}^n$. If h is a smooth real function defined on M, its differential is denoted by Dh or h'; the Hessian Hess $h(x)$ is a bilinear form the value of which is denoted by Hess $h(x) < u, \overline{u} >$, for tangent vectors (at x) u and $\overline{u}$.
For $\beta \in \mathbb{N}^*$, we say that a function is C^β on a closed set F if it is C^β on an open set containing F. Recall also that a real function χ defined on M is said to be convex if for any M-valued geodesic γ, $\chi \circ \gamma$ is convex in the usual sense (if χ is smooth, this is equivalent to require that Hess χ be nonnegative).
For a matrix z with n rows and d columns, ${}^t z$ denotes its transpose,

$$\|z\| = \sqrt{\mathrm{Tr}(z\,{}^t z)} = \sqrt{\sum_{i=1}^{d} |[{}^t z]^i|^2}$$

(Tr is the trace of a square matrix) and $\|z\|_r = \sqrt{\sum_{i=1}^{d} |[{}^t z]^i|_r^2}$ where the columns of z are considered as tangent vectors. The notation $\Psi(x,x') \approx \delta(x,x')^\nu$ means that there is a constant $c > 0$ such that

$$\forall x, x',\ \frac{1}{c}\ \delta(x,x')^\nu \le \Psi(x,x') \le c\ \delta(x,x')^\nu.$$

Throughout the article, we consider an open set O of M, relatively compact in a local chart and an open set $\omega \neq \emptyset$ relatively compact in O, verifying that

• There is a unique geodesic in $\overline{O}$, linking any two points of $\overline{O}$, and depending smoothly on its endpoints;
• $\overline{\omega} = \{\chi \le c\}$, the sublevel set of a smooth convex function χ defined on O. Note

that O will be as well considered as a subset of $\mathbb{R}^n$.
We recall that in the case of a general connection, any point x of M has a neighborhood O for which the first property holds; and when the Levi-Civita connection is used, the first property is also true for regular geodesic balls.

3 Case of a drift f verifying a monotonicity condition

3.1 Assumptions on f

In this section, we explicit the assumptions on the drift f under which we will work; the main difference with [1] and [2] is that we replace the Lipschitz property in x by a monotonicity condition. First, we suppose that f verifies the Lipschitz condition in the variables b and z

$$\begin{aligned}&\exists L>0,\ \forall b,b'\in\mathbb{R}^d,\forall x\in O,\forall z,z'\in\mathcal{L}(\mathbb{R}^{d_w},T_xM),\\&|f(b,x,z)-f(b',x,z')|_r\leq L\Bigg(|b-b'|(1+\|z\|_r+\|z'\|_r)+\|z-z'\|_r\Bigg).\end{aligned}\tag{3.1}$$

Before introducing the monotonicity condition in x, we recall that, in the case of a general connection, Ψ is a smooth, convex and nonnegative function, vanishing on the diagonal only and such that $\Psi\approx\delta^p$. In the case of the Levi-Civita connection, we take $\Psi=\delta^2/2$ (in fact it is equivalent to take $\Psi=\delta$ off the diagonal).
The monotonicity condition is written

$$\begin{aligned}&\exists\nu\in\mathbb{R},\ \forall b\in\mathbb{R}^d,\forall x,x'\in O,\forall z\in\mathcal{L}(\mathbb{R}^{d_w},T_xM),\\&D\Psi(x,x')\cdot\begin{pmatrix}f(b,x,z)\\ f(b,x',\overset{x'}{\underset{x}{\|}}z)\end{pmatrix}\geq\nu\Psi(x,x')(1+\|z\|_r)\end{aligned}\tag{3.2}$$

where

$$\overset{x'}{\underset{x}{\|}}z$$

denotes the parallel transport (defined by the connection) of the d_w columns of the matrix z (considered as tangent vectors) along the unique geodesic between x and x'. This monotonicity condition replaces here the well-known monotonicity condition involving the inner product in an Euclidean space (see e.g. Assumption (4) in [3] or Assumption ($H3$) in [4]). Note that here we have a lower bound on $D\Psi$ (and not an upper bound as in the articles cited above) because in the equation $(M+D)$, the drift f is given with a "plus" sign.

We also need the following uniform boundedness condition

$$\exists L_2>0,\forall b\in\mathbb{R}^d,\forall x\in O,|f(b,x,0)|_r\leq L_2\tag{3.3}$$

and the continuity in the x variable :

$$\forall b\in\mathbb{R}^d,\ \forall z\in\mathcal{L}(\mathbb{R}^{d_w},T_xM),\quad x\mapsto f(b,x,z)\text{ is continuous.}\tag{3.4}$$

3.2 The results

We state now the existence and uniqueness theorem, which generalizes Theorem 1.4.1 of [1] and Theorem 1.3.1 of [2] to our context. We need the following assumption :

$$(H)\quad f \text{ is pointing outward on the boundary of } \overline{\omega}.$$

Theorem 3.2.1 *We consider the BSDE $(M+D)$ with terminal random variable $U \in \overline{\omega} = \{\chi \leq c\}$. We suppose that f verifies conditions (3.1), (3.2), (3.3) and (3.4), and that χ is strictly convex (i.e. Hess χ is positive definite). Moreover for (i) and (ii), f is also supposed to verify (H).*

(i) If f does not depend on z, the BSDE has a unique solution $(X_t, Z_t)_{0\leq t\leq T}$ such that X remains in $\overline{\omega}$.

(ii) If the Levi-Civita connection is used, then the BSDE has yet a unique solution $(X_t, Z_t)_{0\leq t\leq T}$ with X in $\overline{\omega}$ (this is true in particular for regular geodesic balls) .

(iii) In the case of a general connection, each point q of M has a neighborhood $O_q \subset O$ such that, if $\overline{\omega} \subset O_q$ and f verifies hypothesis (H), then the BSDE $(M+D)$ has a unique solution $(X_t, Z_t)_{0\leq t\leq T}$ such that X remains in $\overline{\omega}$. The neighborhood O_q depends on the geometry of the manifold, but not on the drift f.

3.3 Preliminary lemmas

First we recall two results : Proposition 2.2.1 in [2] and Lemma 3.2.1 of [1] (in fact Lemma 3.3.2 below is a straightforward consequence of this lemma and Proposition 3.3.1).

Proposition 3.3.1 *Here O is considered, via a system of local coordinates, as an open subset of $\mathbb{R}^n$. There is a $C > 0$ such that for every $(x, x') \in O \times O$ and $(z, z') \in T_xM \times T_{x'}M$, we have*

$$\left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r \leq C\left(|z - z'| + \delta(x, x')(|z| + |z'|)\right)$$

and

$$|z - z'| \leq C \left(\left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r + \delta(x, x')(|z|_r + |z'|_r) \right). \tag{3.5}$$

Lemma 3.3.2 *Suppose that $\Psi(x, x') \approx \delta(x, x')^p$ on $O \times O$ where p is an even positive integer (since Ψ is smooth). Then there is $C > 0$ such that, for all vectors $(z, z') \in T_xM \times T_{x'}M$,*

$$\left| D\Psi(x, x') \cdot \begin{pmatrix} z \\ z' \end{pmatrix} \right| \leq C\delta(x, x')^{p-1} \left(\delta(x, x')(|z|_r + |z'|_r) + |\overset{x'}{\underset{x}{\|}} z - z'|_r \right). \tag{3.6}$$

Then we have the following lemma, which is in fact the main argument in order to prove uniqueness. For notational convenience, we denote $f(b, x, z)$ by f and $f(b, x', z')$ by f'.

Lemma 3.3.3 *There is $\hat{C} > 0$ such that, for all b in $\mathbb{R}^d$, x, x' in O, $z \in \mathcal{L}(\mathbb{R}^{d_w}, T_x M)$ and $z' \in \mathcal{L}(\mathbb{R}^{d_w}, T_{x'} M)$,*

$$D\Psi(x,x') \cdot \begin{pmatrix} f \\ f' \end{pmatrix} \geq -\hat{C}\delta(x,x')^{p-1} \left(\left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r + \delta(x,x')(1 + \|z\|_r + \|z'\|_r) \right). \tag{3.7}$$

Proof. Note that we use the same b. We have

$$\begin{aligned} D\Psi(x,x') \cdot \begin{pmatrix} f \\ f' \end{pmatrix} &= D\Psi(x,x') \cdot \begin{pmatrix} f(b,x,z) \\ f(b,x',\overset{x'}{\underset{x}{\|}} z) \end{pmatrix} \\ &\quad + D\Psi(x,x') \cdot \begin{pmatrix} 0 \\ f(b,x',z') - f(b,x',\overset{x'}{\underset{x}{\|}} z) \end{pmatrix}. \\ &\geq \nu\Psi(x,x')(1 + \|z\|) \\ &\quad - C\delta(x,x')^{p-1} \left(\left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r + \delta(x,x')(1 + \|z\|_r + \|z'\|_r) \right). \end{aligned}$$

In the last inequality, we have used the monotonicity assumption (3.2) for the first term, and for the second term the Lipschitz assumption (3.1) and Lemma 3.3.2 above. This gives the result since $\Psi \approx \delta^p$. □

To end this part, we give the following result, which means that f has at most a linear growth in the variable z. It will be useful in the sequel.

Lemma 3.3.4 *Under the above assumptions* (3.1) *and* (3.3)*, there is a constant C such that*

$$\forall\ b, x, z, \quad |f(b,x,z)|_r \leq C(\|z\|_r + 1).$$

Proof. It is a straightforward consequence of the two assumptions (3.1) and (3.3) :

$$\begin{aligned} \forall\ b, x, z, \quad |f(b,x,z)|_r &\leq |f(b,x,z) - f(b,x,0)|_r + |f(b,x,0)|_r \\ &\leq L\|z\|_r + L_2. \end{aligned}$$

This gives the result. □

4 The uniqueness property

We recall first the method used in [1] and in [2], then we will sketch the main changes to make under our new assumptions.

4.1 General results

Consider two solutions $(X_t, Z_t)_{0\le t\le T}$ and $(X'_t, Z'_t)_{0\le t\le T}$ of $(M+D)$ such that X and X' remain in $\overline{\omega}$ and $X_T = Y_T = U$ (we will say sometimes "$\overline{\omega}$-valued solutions of $(M+D)$"). Let

$$\tilde{X}_s = (X_s, X'_s) \quad \text{and } \tilde{Z}_s = \begin{pmatrix} Z_s \\ Z'_s \end{pmatrix}.$$

To prove uniqueness, the idea is to show that the process $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$ where

$$A_t = \lambda t + \mu \int_0^t (\|Z_s\|_r^\alpha + \|Z'_s\|_r^\alpha)ds,$$

is a submartingale for appropriate nonnegative constants λ, μ and α, and a suitable function Ψ, smooth on $O \times O$ (of course in general we will have to prove some integrability properties).

If we apply Itô's formula to (S_t), we get

$$\begin{aligned} e^{A_t}\Psi(\tilde{X}_t) - \Psi(\tilde{X}_0) &= \int_0^t e^{A_s} d(\Psi(\tilde{X}_s)) + \int_0^t e^{A_s}(\lambda + \mu(\|Z_s\|_r^\alpha + \|Z'_s\|_r^\alpha))\Psi(\tilde{X}_s)ds \\ &= \int_0^t e^{A_s} D\Psi(\tilde{X}_s)\left(\tilde{Z}_s dW_s\right) \\ &\quad + \frac{1}{2}\int_0^t e^{A_s}\left(\sum_{i=1}^{d_W} {}^t[{}^t\tilde{Z}_s]^i \text{Hess } \Psi(\tilde{X}_s)[{}^t\tilde{Z}_s]^i\right) ds \\ &\quad + \int_0^t e^{A_s} D\Psi(\tilde{X}_s)\begin{pmatrix} f(B_s^y, X_s, Z_s) \\ f(B_s^y, X'_s, Z'_s)\end{pmatrix} ds \\ &\quad + \int_0^t e^{A_s}\Psi(\tilde{X}_s)(\lambda + \mu(\|Z_s\|_r^\alpha + \|Z'_s\|_r^\alpha))ds. \end{aligned} \tag{4.1}$$

Thus to prove the submartingale property, we need to show the nonnegativity of the sum

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{d_W} {}^t[{}^t\tilde{Z}_t]^i \text{Hess } \Psi(\tilde{X}_t)[{}^t\tilde{Z}_t]^i \quad &+ \quad D\Psi(\tilde{X}_t)\begin{pmatrix} f(B_t^y, X_t, Z_t) \\ f(B_t^y, X'_t, Z'_t)\end{pmatrix} \\ &+ \quad (\lambda + \mu(\|Z_t\|_r^\alpha + \|Z'_t\|_r^\alpha))\Psi(\tilde{X}_t). \end{aligned} \tag{4.2}$$

4.2 Sketch of the proof

Case of a drift independent of z

In this case, the result is straightforward; indeed, from Lemma 3.3.3, there is $C > 0$ such that, for all b in $\mathbb{R}^d$, x, x' in $\overline{\omega}$,

$$D\Psi(x,x')\cdot\begin{pmatrix} f(b,x) \\ f(b,x')\end{pmatrix} \ge -C\delta(x,x')^p \ge -\tilde{C}\Psi(x,x').$$

Therefore in order to make the sum (4.2) nonnegative, it is sufficient to take λ larger than $|\tilde{C}|$ and $\mu = 0$.

Uniqueness for drifts depending on z

As we remarked above, uniqueness is a consequence of the nonnegativity of the sum (4.2). Of course the estimates on the Hessian term which are given in [1] and [2] are still valid; thus we have only to verify that the same kind of estimates hold for the term involving f in (4.2). Moreover, we need to keep the integrability conditions which make integrable the process $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$.

We begin with the integrability condition; in fact it is based upon Lemma 3.4.2 of [1] which we recall now.

Lemma 4.2.1 *Suppose that we are given a positive constant α and a C^2 function ϕ on $\overline{\omega}$ satisfying $C_{min} \leq \phi(x) \leq C_{max}$ for some positive C_{min} and C_{max}. Suppose moreover that* Hess $\phi + 2\alpha\phi \leq 0$ *on $\overline{\omega}$; this means that*

$$\text{Hess } \phi(x) < u, u > +2\alpha\phi(x)|u|_r^2 \leq 0. \tag{4.3}$$

Then, for every $\varepsilon > 0$, any $\overline{\omega}$-valued solution of $(M + D)$ belongs to $(\mathcal{E}_{\alpha-\varepsilon})$.

An accurate examination of the proof given in [1] shows that the only hypothesis needed on the drift f is the linear growth in z, and this is yet verified, according to Lemma 3.3.4.

Therefore it remains to verify that the estimates on $D\Psi(x, x')(f, f')$ are still valid. We just check that this is true in the three cases dealt with in the two articles [1] and [2].

In the case of a Levi-Civita connection, two different approaches were developed : (1) On the one hand for nonpositive sectional curvatures, the function Ψ is just $\delta^2/2$ and Lemma 3.3.3 gives the estimate

$$D\Psi(\tilde{x}) \begin{pmatrix} f(b, x, z) \\ f(b, x', z') \end{pmatrix} \geq -C\delta^2(x, x')(1 + \|z\|_r + \|z'\|_r) - \frac{1}{4}\left\| \mathop{\|}_{x}^{x'} z - z' \right\|_r^2. \tag{4.4}$$

(2) On the other hand, if positive sectional curvatures are allowed (bounded by $K > 0$), we defined in [2]

$$\Psi(x, x') = \Psi_a(x, x') = \sin^a\left(\sqrt{K}\frac{\delta(x, x')}{2}\right)$$

and therefore, for $z \in \mathcal{L}(\mathbb{R}^{d_w}, T_xM)$ and $z' \in \mathcal{L}(\mathbb{R}^{d_w}, T_{x'}M)$,

$$\begin{aligned}
D\Psi(\tilde{x}) \cdot \begin{pmatrix} f(b,x,z) \\ f(b,x',z') \end{pmatrix} &= a \sin^{a-1}(y) \cos y \cdot \frac{\sqrt{K}}{2} \delta'(\tilde{x}) \cdot \begin{pmatrix} f(b,x,z) \\ f(b,x',z') \end{pmatrix} \\
&\geq -a \sin^{a-1}(y) \frac{\sqrt{K}}{2} \\
&\qquad \times L_0 \left(\delta(\tilde{x})(1 + \|z\|_r + \|z'\|_r) + \left\| \mathop{\|}\limits_x^{x'} z - z' \right\|_r \right) \\
&\geq -a\frac{\pi}{2} \Psi(\tilde{x}) L (1 + \|z\|_r + \|z'\|_r) - C_1 \sin^{a-1} y \left\| \mathop{\|}\limits_x^{x'} z - z' \right\|_r \\
&\geq -C\Psi(\tilde{x}) - \frac{e-1}{2}\frac{K}{4}\Psi(\tilde{x})(\|z\|_r^2 + \|z'\|_r^2) \\
&\qquad -\frac{\alpha}{2} \sin^{a-2} y \left\| \mathop{\|}\limits_x^{x'} z - z' \right\|_r^2 . \qquad (4.5)
\end{aligned}$$

for appropriate constants a, e and α (which are in fact determined according to the estimates upon Hess Ψ in the same way as in [2]), and positive constants C_1 and C. Note that the first inequality is a consequence of the monotonicity condition (3.2) with $\Psi = \delta$ instead of δ^2 (we have already remarked that we have equivalence off the diagonal).

(3) In the case of a more general connection, Lemma 3.3.3 gives, for any $\varepsilon > 0$,

$$\begin{aligned}
D\Psi \cdot \begin{pmatrix} f(b,x,z) \\ f(b,x',z') \end{pmatrix} &\geq -C_\varepsilon \Psi(x,x')(1 + \|z\|_r + \|z'\|_r) \\
&\qquad -\varepsilon \delta^{p-2}(x,x') \left\| \mathop{\|}\limits_x^{x'} z - z' \right\|_r^2 . \qquad (4.6)
\end{aligned}$$

It turns out that the estimate (4.4) (resp. (4.5), resp. (4.6)) is exactly what is needed on the term $D\Psi(f, f')$ in order to get the nonnegativity of the sum (4.2); thus it suffices to replicate the proof of Lemma 3.4.5 in [1] (resp. Proposition 3.3.2 in [2], resp. Theorem 3.2.2 in [2]). This concludes the uniqueness part.

5 Existence property

First we give the framework of the proof.

Suppose that $c > 0$ and χ reaches its infimum at $p \in \omega$ with $\chi(p) = 0$. Then consider the following mapping, defined on a normal open neighborhood of $\overline{\omega}$ centered at p (so it is in particular a neighborhood of 0 in $\mathbb{R}^n$)

$$y \mapsto \frac{\sqrt{c_2}\, y}{\sqrt{c_2 \|y\|^2 + c - \chi(y)}};$$

for $c_2 > 0$ small enough, it is a diffeomorphism from an open set O_1 (relatively compact in O and containing $\overline{\omega}$) onto an open neighborhood N of $\overline{B(0,1)}$, such that

$\overline{\varpi}$ is sent onto $\overline{B(0,1)}$ (in fact, it is sufficient to take $c_2 \leq \frac{1}{2}\lambda_\chi$, where λ_χ denotes the (positive) infimum on O_1 of the eingenvalues of Hess χ). Using this diffeomorphism, we can work in a local chart O (take $O := N$) such that $\overline{\varpi} = \overline{B(0,1)}$ and $\chi(0) = 0$. We will mainly use the Euclidean norm in this section; note in particular that (3.1), (3.2) and (3.3) remain true if the Riemannian norm is replaced by the Euclidean one (changing accordingly the constants).
Remark that hypothesis (H) means, in the new coordinates, that the radial component $f^{\mathrm{rad}}(b,x,z)$ of $f(b,x,z)$ is nonnegative for $x \in \partial\overline{\varpi}$.

The goal in this part is to approximate f by Lipschitz functions for which the existence theorems of [1] and of [2] apply. However, unlike in these articles, we cannot use the convolution product directly; indeed, we don't have precise estimates for f in a neighborhood of $\overline{\varpi}$, therefore we can't prove that these approximations be pointing outward on $\partial\overline{\varpi}$. In order to give such estimates, we need first to introduce new functions f_k with which we shall use the convolution product.

First step. We truncate the growth of the drift f, both in b and z, in the following way. Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be a smooth nonincreasing function with

$$\begin{cases} \phi(u) = 1 & \text{if} \ \ u \leq 1 \\ \phi(u) = 0 & \text{if} \ \ u \geq 2; \end{cases}$$

then put $\phi_k(u) = \phi(u - (k-1))$ for any positive integer k. Note that $0 \leq \phi_k \leq 1$, with $\phi_k(u) = 1$ if $u \leq k$ and $\phi_k(u) = 0$ if $u \geq k+1$. Let

$$f_k(b,x,z) = f(b,x,z)\phi_k(|b|)\phi_k(\|z\|); \tag{5.1}$$

in particular, f_k tends to f when k tends to infinity. We have the

Lemma 5.0.2 *There is a constant C, independent of k, such that*

$$(i) \qquad \forall x, x' \in O, \forall z \in \mathbb{R}^{nd_w}, \quad \left| \phi_k\left(\left\| \mathop{\|}_{x}^{x'} z \right\| \right) - \phi_k(\|z\|) \right| \leq Ck\delta(x,x');$$

$$(ii) \qquad \forall x \in O, \forall z, z' \in \mathbb{R}^{nd_w}, \forall b \in \mathbb{R}^d,$$
$$|f(b,x,z')| \cdot |\phi_k(\|z'\|) - \phi_k(\|z\|)| \leq C(1 + k \wedge \|z'\|)\|z - z'\|.$$

Proof. The function ϕ_k verifies $|\phi_k(u) - \phi_k(u')| \leq C_0|u-u'|(1_{\{u \leq k+1\}} + 1_{\{u' \leq k+1\}})$; using Proposition 3.3.1, we get

$$\begin{aligned} \left| \phi_k\left(\left\| \mathop{\|}_{x}^{x'} z \right\| \right) - \phi_k(\|z\|) \right| &\leq C_0 \left\| \mathop{\|}_{x}^{x'} z - z \right\| \left(1_{\{\|z\| \leq k+1\}} + 1_{\{\| \mathop{\|}_{x}^{x'} z \| \leq k+1\}}\right) \\ &\leq C_1 \left(\delta(x,x')\|z\| 1_{\{\|z\| \leq k+1\}} + \delta(x,x') \| \mathop{\|}_{x}^{x'} z \| 1_{\{\| \mathop{\|}_{x}^{x'} z\| \leq k+1\}} \right) \\ &\leq C(k+1)\delta(x,x'), \end{aligned}$$

hence (i).

For (ii), as ϕ_k is Lipschitz (with a constant independent of k), we have

$$|f(b,x,z')| \cdot |\phi_k(\|z'\|) - \phi_k(\|z\|)| \leq C(1 + \|z'\|)\|z - z'\|.$$

As f has a linear growth in z, it is thus sufficient to prove

$$\|z'\| \cdot |\phi_k(\|z'\|) - \phi_k(\|z\|)| \leq C(1 + k)\|z - z'\|.$$

The only case for which we have to be careful is $z' \geq k+1$ and $z < k+1$. We have in this case

$$\begin{array}{rcl} \|z'\| \cdot |\phi_k(\|z'\|) - \phi_k(\|z\|)| & \leq & (\|z' - z\| + \|z\|) \cdot |\phi_k(\|z'\|) - \phi_k(\|z\|)| \\ & \leq & 2\|z' - z\| + (k+1)\|z' - z\| \end{array}$$

and that completes the proof. $\square$

Proposition 5.0.3 *We have the following properties for the new drift f_k :*
(i) It verifies a Lipschitz property in b and z in the following sense :

$$\exists \hat{L} > 0,\ \forall b, b' \in \mathbb{R}^d, \forall x \in O, \forall z, z' \in \mathbb{R}^{nd_w},$$

$$|f_k(b', x, z') - f_k(b, x, z)| \leq \hat{L}\Bigg((|b - b'| + \|z - z'\|) \times (1 + k \wedge \|z\| + k \wedge \|z'\|) \Bigg)$$

(ii) It verifies a monotonicity condition like (3.2)*, but now with a constant depending on k :*

$$\exists C > 0,\ \forall b \in \mathbb{R}^d, \forall x, x' \in O, \forall z \in \mathbb{R}^{nd_w},$$
$$D\Psi(x, x') \cdot \begin{pmatrix} f_k(b, x, z) \\ f_k(b, x', \overset{x'}{\underset{x}{\|}} z) \end{pmatrix} \geq -Ck\Psi(x, x')(1 + \|z\|).$$

(iii) When $z = 0$, it is uniformly bounded as in (3.3) *(independently of k).*

(iv) It is continuous in x in the sense of (3.4)*.*

(v) It is pointing outward on the boundary of $\overline{\omega}$.

(vi) Recall from the beginning of this section that $\overline{O}_1$ is a compact set verifying $\overline{\omega} \subset O_1 \subset \overline{O}_1 \subset O$. Then f_k is uniformly continuous in the x variable, i.e.

$$\begin{array}{c} \forall \varepsilon > 0, \exists \eta_k > 0 \quad s.t.\ \forall (b, x, z) \in \mathbb{R}^d \times \overline{O}_1 \times \mathbb{R}^{nd_w}, \forall v \quad s.t.\ |v| \leq \eta_k, \\ |f_k(b, x, z) - f_k(b, x + v, z)| \leq \varepsilon. \end{array}$$

Proof. Using that $\phi_k(u) \le 1$, we have

$$
\begin{aligned}
|f_k(b',x,z') - f_k(b,x,z)| &\le |f(b',x,z')\phi_k(|b'|) - f(b,x,z')\phi_k(|b|)| \cdot \phi_k(\|z'\|) \\
&\quad +|f(b,x,z')\phi_k(\|z'\|) - f(b,x,z)\phi_k(\|z\|)| \\
&\le |f(b',x,z')| \cdot |\phi_k(|b'|) - \phi_k(|b|)| \cdot \phi_k(\|z'\|) \\
&\quad +|f(b',x,z') - f(b,x,z')| \cdot \phi_k(\|z'\|) \\
&\quad +|f(b,x,z')| \cdot |\phi_k(\|z'\|) - \phi_k(\|z\|)| \\
&\quad +|f(b,x,z') - f(b,x,z)| \cdot \phi_k(\|z\|) \\
&\le C_1(1+\|z'\|)|b-b'|1_{\|z'\|\le k+1} \\
&\quad +C_2(1+k\wedge\|z'\|)\|z-z'\| \\
&\quad +C_3\|z-z'\|.
\end{aligned}
$$

In the last inequality we have used (3.1), Lemma 3.3.4 (for the Euclidean norm) and (ii) of Lemma 5.0.2. Since $(1+\|z'\|)1_{\|z'\|\le k+1} \le Ck\wedge\|z'\|$, this proves (i).

The properties (iii), (iv) and (v) follow obviously from the same properties for f. Moreover, (vi) is just a consequence of the continuity of f on the compact set $\overline{B(0,k+1)} \times \overline{O}_1 \times \overline{B(0,k+1)}$ (recall that if $b \notin \overline{B(0,k+1)}$ or $z \notin \overline{B(0,k+1)}$, then $f_k(b,x,z) = 0$).

For (ii), we write f_k (resp f) for $f_k(b,x,z)$ (resp. $f(b,x,z)$) and f'_k (resp f') for $f_k(b,x',\overset{x'}{\underset{x}{\|}}z)$ (resp. $f(b,x',\overset{x'}{\underset{x}{\|}}z)$) and we have

$$
\begin{aligned}
D\Psi\cdot\begin{pmatrix} f_k \\ f'_k \end{pmatrix} &= D\Psi\cdot\begin{pmatrix} \phi_k(\|z\|)\phi_k(|b|)f \\ \phi_k(\|\overset{x'}{\underset{x}{\|}}z\|)\phi_k(|b|)f' \end{pmatrix} \\
&= D\Psi\cdot\begin{pmatrix} f \\ f' \end{pmatrix}\phi_k(|b|)\phi_k(\|z\|) \\
&+ D\Psi\cdot\begin{pmatrix} 0 \\ (\phi_k(\|\overset{x'}{\underset{x}{\|}}z\|) - \phi_k(\|z\|))f' \end{pmatrix}\phi_k(|b|) \\
&\ge (\nu\wedge 0)\Psi(x,x')(1+\|z\|) - C_1k\Psi(x,x')(1+\|z\|) \\
&\ge -Ck\Psi(x,x')(1+\|z\|).
\end{aligned}
$$

In the first inequality, we have used for the first term the monotonicity assumption (3.2) on f, and for the second term the Euclidean form of Lemma 3.3.2, (i) in Lemma 5.0.2 and the linear growth of f in z (Lemma 3.3.4). Note that the constant $C > 0$ at the end does not depend on k. Then we have (ii). □

The interest of the new drifts f_k lies in condition (vi); indeed, this gives some information about the behavior of f_k around the boundary $\partial\overline{\omega}$. Then now we can construct new approximations which will verify Lipschitz conditions in all the variables.

Second step. Extend each mapping f_k to $\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{nd_w}$ by putting $f_k(b,x,z) = 0$ if $x \notin O$ and define (on $\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{nd_w}$) the convolution product for $l \in \mathbb{N}^*$

$f_{k,l} = f_k * \rho_l$:

$$f_{k,l}(b,x,z) = \int_{\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{nd_w}} f_k((b,x,z)-(\beta,y,w))\rho_l(\beta,y,w)d(\beta,y,w)$$

where $\rho_l(b,x,z) = l\rho(l\|(b,x,z)\|)$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a bump function (i.e. a smooth function with $\rho'(0)=0$, $\rho = 0$ outside $[0;1]$ and $\int_{\mathbb{R}_+}\rho(u)du = 1$).

Lemma 5.0.4 *The following holds for the mappings $f_{k,l}$:*

(i) It verifies the Lipschitz property

$$\exists L(k,l) > 0,\ \forall b,b' \in \mathbb{R}^d, \forall x,x' \in \overline{O}_1, \forall z,z' \in \mathbb{R}^{nd_w},$$

$$|f_{k,l}(b,x,z) - f_{k,l}(b',x',z')| \le L(k,l)\Bigg((|b-b'|+|x-x'|)(1+\|z\|+\|z'\|)+\|z-z'\| \Bigg). \tag{5.2}$$

(ii) When $z=0$, it is uniformly bounded as in (3.3)*, and the bound does not depend on k,l.*

Proof. As soon as $dist(O_1,O) > 1/l_0 \ge 1/l$, $\rho_l(\beta,y,w) = 0$ if $|y| \ge dist(O_1,O)$; so the integrand vanishes if $x-y \notin O$ and we can use the properties of f_k on O.
It is then obvious that $f_{k,l}$ is Lipschitz (in the sense of (3.1)) in the variables b and z (and the Lipschitz constant can be taken the same as the one for f_k). Then the only point to check is for the x variable. But one has

$$f_{k,l}(b,x,z) = \int_{\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{nd_w}} \rho_l((b,x,z)-(\beta,y,w))f_k(\beta,y,w)d(\beta,y,w);$$

since $x \mapsto \rho_l((b,x,z)-(\beta,y,w))$ is a smooth function with bounded partial derivatives, we get

$$\begin{aligned} |D_x f_{k,l}(b,\cdot,z)| &\le C_1(k,l)\int_{\|(b,x,z)-(\beta,y,w)\|\le 1/l} |f_k(\beta,y,w)|d(\beta,y,w) \\ &\le C_{k,l}(1+\|z\|) \end{aligned}$$

where the last inequality results from the linear growth in z of f_k (using Lemma 3.3.4). Thus $x \mapsto f_{k,l}(b,x,z)$ verifies a Lipschitz condition with a constant growing linearly with z, as was to be proved.

Finally, assumption (ii) results from the same property for f_k ((iii) in Proposition 5.0.3) and the Lipschitz property (3.1) for f :

$$\begin{aligned} |f_{k,l}(b,x,0)| &\le \int_{\|(\beta,y,w)\|\le 1/l} |f_k(b-\beta,x-y,-w)|\rho_l(\beta,y,w)d(\beta,y,w) \\ &\le \int_{\|(\beta,y,w)\|\le 1/l} C_0\rho_l(\beta,y,w)d(\beta,y,w) = C_0 \end{aligned}$$

using the definition of ρ_l. $\square$

Now the mappings $f_{k,l}$ are not necessarily pointing outward on the boundary $\partial\overline{\omega}$; in order to get this property, we introduce the new drift

$$g_{k,l}(b,x,z) = f_{k,l}(b,x,z) + (\varepsilon_{k,l} + \frac{A}{l}(1 + \|z\|))x,$$

where $\varepsilon_{k,l}$ is the constant associated with $\eta = 1/l$ in the uniform continuity assumption (vi) of Proposition 5.0.3 and A is a constant which is chosen below.

Proposition 5.0.5 *The mapping $g_{k,l}$ verifies properties (i) and (ii) in Lemma 5.0.4 above. Moreover, there is a positive constant A, depending on k* **but not on** *l, such that $g_{k,l}$ is pointing outward on the boundary $\partial\overline{\omega}$.*

Therefore, from Theorem 1.4.1 in [1] and Theorem 1.3.1 in [2], the BSDE $(M + D)$ with drift $g_{k,l}$ has a unique solution $(X_{k,l}, Z_{k,l})$ with $X_{k,l} \in \overline{\omega}$.

Proof. That $g_{k,l}$ verifies (i) and (ii) in Lemma 5.0.4 above is obvious. Now let us prove the last property; consider $x \in \partial\overline{\omega} = S(0,1)$. If we put $E := |f_{k,l}(b,x,z) - f_k(b,x,z)|$ then

$$\begin{aligned}
E &= \left| \int (f_k(b-\beta, x-y, z-w) - f_k(b,x,z))\rho_l(\beta,y,w)d(\beta,y,w) \right| \\
&\leq \int |f_k(b-\beta, x-y, z-w) - f_k(b-\beta, x, z-w)|\, \rho_l(\beta,y,w)d(\beta,y,w) \\
&\quad + \int |f_k(b-\beta, x, z-w) - f_k(b,x,z)|\, \rho_l(\beta,y,w)d(\beta,y,w) \\
&\leq \varepsilon_{k,l} + \frac{C}{l}(1 + \|z\|)
\end{aligned}$$

where the last inequality is a consequence of (i) and (vi) in Proposition 5.0.3 (note that C depends neither on k nor on l).

Since f_k is pointing outward on $\partial\overline{\omega}$, this shows that the following lower bound on the radial part of $f_{k,l}$ holds on $\partial\overline{\omega}$:

$$[f_{k,l}(b,x,z)]^{\mathrm{rad}} \geq -\varepsilon_{k,l} - \frac{C}{l}(1 + \|z\|).$$

Now if we choose $A := C + 1$, which is independent of k and l, then the mapping $g_{k,l}$ is also pointing outward on $\partial\overline{\omega}$. This completes the proof of the proposition. $\square$

Third step. Now it remains to pass through the limit; first for k fixed and $l \to \infty$, then for $k \to \infty$. Note that the terminal value $U \in \overline{\omega}$ is fixed independently of k, l.

First let k fixed and $l \to \infty$. Let (X^l, Z^l) be the (unique) solution of the equation $(M + D)$ with the drift $g_{k,l}$ defined above. In general, we will drop the superscript k in the notations, for simplicity.

For $l, l' \in \mathbb{N}^*$, we put $\tilde{X}_t^{l,l'} := (X_t^l, X_t^{l'})$, $\tilde{Z}_t^{l,l'} := (Z_t^l, Z_t^{l'})$ and $V_t^l := (B_t^y, X_t^l, Z_t^l)$. We apply Iô's formula between t and T to $\hat{\Psi}$ defined by

$$\begin{cases} \hat{\Psi} = \Psi \approx \delta^2 & \text{if a general connection is used;} \\ \hat{\Psi} = \delta^2 & \text{in the case of the Levi-Civita connection.} \end{cases}$$

We get

$$\begin{aligned} -\hat{\Psi}(\tilde{X}_t^{l,l'}) &= \int_t^T D\hat{\Psi}(\tilde{X}_s^{l,l'}) \left(\tilde{Z}_s^{l,l'} dW_s \right) \\ &\quad + \frac{1}{2} \int_t^T \left(\sum_{i=1}^{d_W} {}^t[{}^t\tilde{Z}_s^{l,l'}]^i \text{Hess } \hat{\Psi}(\tilde{X}_s^{l,l'}) [{}^t\tilde{Z}_s^{l,l'}]^i \right) ds \\ &\quad + \int_t^T D\hat{\Psi}(\tilde{X}_s^{l,l'}) \begin{pmatrix} g_{k,l}(V_s^l) \\ g_{k,l'}(V_s^{l'}) \end{pmatrix} ds \end{aligned} \tag{5.3}$$

In fact this does not include the case of a drift f which does not depend on z; for this simpler case, see the remark at the end of this section.

The first term on the right is a martingale; the term involving the Hessian is bounded below by

$$\hat{\alpha} \int_t^T \left\| \mathop{\|}_{X_s^l}^{X_s^{l'}} Z_s^l - Z_s^{l'} \right\|_r^2 ds - \hat{\beta} \int_t^T \hat{\Psi}(\tilde{X}_s^{l,l'})(\|Z_s^l\|_r^2 + \|Z_s^{l'}\|_r^2) ds,$$

or

$$\alpha \int_t^T \left\| Z_s^l - Z_s^{l'} \right\|^2 ds - \beta \int_t^T \hat{\Psi}(\tilde{X}_s^{l,l'})(\|Z_s^l\|^2 + \|Z_s^{l'}\|^2) ds, \tag{5.4}$$

for positive constants α and β. The first bound results from Proposition 2.3.2 in [1] if sectional curvatures are nonpositive (in this case $\beta = 0$) and from estimate (4.2) in [2] in the other cases. The second bound is then a consequence of Proposition 3.3.1.

The last term involving $g_{k,l}$ and $g_{k,l'}$ can be rewritten as

$$A_{l,l'} + \int_t^T D\hat{\Psi}(\tilde{X}_s^{l,l'}) \begin{pmatrix} f_k(V_s^l) \\ f_k(V_s^{l'}) \end{pmatrix} ds$$

where

$$A_{l,l'} := \int_t^T D\hat{\Psi}(\tilde{X}_s^{l,l'}) \begin{pmatrix} (g_{k,l} - f_k)(V_s^l) \\ (g_{k,l'} - f_k)(V_s^{l'}) \end{pmatrix} ds. \tag{5.5}$$

Using (ii) in Proposition 5.0.3, we have

$$\int_t^T D\hat{\Psi}(\tilde{X}_s^{l,l'}) \begin{pmatrix} f_k(V_s^l) \\ f_k(V_s^{l'}) \end{pmatrix} ds \geq -C \cdot k \int_t^T \hat{\Psi}(\tilde{X}_s^{l,l'})(1 + \|Z_s^l\| + \|Z_s^{l'}\|) ds \tag{5.6}$$

for a constant C independent of l. Now using (5.4) and (5.6), we get by taking the expectation in (5.3)

$$\begin{aligned}\mathbb{E}(\hat{\Psi}(\tilde{X}_t^{l,l'})) + \alpha\mathbb{E}\int_t^T \left\|Z_s^l - Z_s^{l'}\right\|^2 ds &\leq C_k^1\Bigg(\int_t^T \mathbb{E}\Big(\hat{\Psi}(\tilde{X}_s^{l,l'})(1 + \|Z_s^l\|_r^2 \\ &\qquad + \|Z_s^{l'}\|_r^2)\Big)ds + \mathbb{E}|A_{l,l'}|\Bigg) \\ &\leq C_k^2\left(\int_t^T \mathbb{E}\left(\hat{\Psi}(\tilde{X}_s^{l,l'})\right)ds + \mathbb{E}|A_{l,l'}|\right)\end{aligned} \tag{5.7}$$

where the last inequality is obtained using Hölder's inequality, noting the uniform exponential integrability of the Z-processes (Lemma 4.2.1) and that $\hat{\Psi}^2 \leq \hat{\Psi}$ on the compact set $\overline{\omega} \times \overline{\omega}$. An application of Gronwall's Lemma gives

$$\mathbb{E}(\hat{\Psi}(\tilde{X}_t^{l,l'})) \leq C_k\mathbb{E}|A_{l,l'}|. \tag{5.8}$$

Lemma 5.0.6 *The expectation* $\mathbb{E}|A_{l,l'}|$*, where* $A_{l,l'}$ *is defined by* (5.5)*, tends to zero as* l, l' *tend to infinity.*

Proof. Since $\hat{\Psi} \approx \delta^2$ and $g_{k,l}(b,x,z) = f_{k,l}(b,x,z) + (\varepsilon_{k,l} + \frac{A}{l}(1 + \|z\|))x$, we have

$$\begin{aligned}\mathbb{E}|A_{l,l'}| &\leq \int_0^T \mathbb{E}\left|D\hat{\Psi}(\tilde{X}_s^{l,l'})\begin{pmatrix}(f_{k,l} - f_k)(V_s^l) \\ (f_{k,l'} - f_k)(V_s^{l'})\end{pmatrix}\right|ds \\ &\quad + \int_0^T \mathbb{E}\left|D\hat{\Psi}(\tilde{X}_s^{l,l'})\begin{pmatrix}(\varepsilon_{k,l} + \frac{A}{l}(1 + \|Z_s^l\|))X_s^l \\ (\varepsilon_{k,l'} + \frac{A}{l'}(1 + \|Z_s^{l'}\|))X_s^{l'}\end{pmatrix}\right|ds \\ &\leq \int_0^T \mathbb{E}\left|D\hat{\Psi}(\tilde{X}_s^{l,l'})\begin{pmatrix}(f_{k,l} - f_k)(V_s^l) \\ (f_{k,l'} - f_k)(V_s^{l'})\end{pmatrix}\right|ds \\ &\quad + C\int_0^T \mathbb{E}\left(\delta(\tilde{X}_s^{l,l'})(\varepsilon_{k,l} + \varepsilon_{k,l'} + (\frac{A}{l} + \frac{A}{l'})(1 + \|Z_s^l\| + \|Z_s^{l'}\|))\right)ds;\end{aligned}$$

The last inequality is obtained with Lemma 3.3.2; the first term on the right tends to zero because $f_{k,l}$ tends uniformly to f_k as l tends to infinity (this results from the properties of convolution and the uniform continuity of f_k). The second term tends to zero because the integral is bounded independently of l (using the uniform exponential integrability condition of Lemma 4.2.1) and $\varepsilon_{k,l}$ tends to zero as l tends to infinity (see (vi) in Proposition 5.0.3). □

As a consequence, as l, l' tend to infinity, $\mathbb{E}\hat{\Psi}(\tilde{X}_s^{l,l'}) \to 0$ (in fact we have obviously $\mathbb{E}\int_0^T \hat{\Psi}(\tilde{X}_s^{l,l'})ds \to 0$), which shows that the sequence of processes X^l is Cauchy in $L^2(\Omega \times [0;T])$ thus converges to a process X^k. From (5.7), we deduce that the sequence of processes Z^l is also Cauchy in $L^2(\Omega \times [0;T])$ and thus converges to a process Z^k as l tends to infinity.

At the end, the pair (X^k, Z^k) solves the equation $(M+D)$ with the drift f_k; passing through the limit is an easy adaptation of the second step in the proof of Proposition 4.1.4 in [1], using once again the uniform continuity of f_k and the uniform convergence of $f_{k,l}$ to f_k. We recall briefly the proof for the sake of completeness.

For k fixed, let us show that the following expectation tends to zero as l tends to ∞ :

$$\begin{aligned}\mathbb{E}\bigg|U \quad - \quad & \int_t^T Z_s^k dW_s - \int_t^T \left(-\frac{1}{2}\Gamma_{ij}(X_s^k)(\left[Z_s^k\right]^i \mid \left[Z_s^k\right]^j) + f_k(B_s^y, X_s^k, Z_s^k)\right) ds \\ + \quad & \int_t^T Z_s^l dW_s + \int_t^T \left(-\frac{1}{2}\Gamma_{ij}(X_s^l)(\left[Z_s^l\right]^i \mid \left[Z_s^l\right]^j) + f_{k,l}(B_s^y, X_s^l, Z_s^l)\right) ds\bigg|.\end{aligned}$$

This expectation is bounded above by

$$\begin{aligned} & \mathbb{E}\left(\int_0^T \|Z_s^l - Z_s^k\|^2 ds\right)^{\frac{1}{2}} \\ + \quad & \mathbb{E}\left(\int_0^T |\Gamma_{ij}(X_s^k) - \Gamma_{ij}(X_s^l)| \cdot |(\left[Z_s^k\right]^i \mid \left[Z_s^k\right]^j)| ds\right) \\ + \quad & \mathbb{E}\left(\int_0^T |\Gamma_{ij}(X_s^l)| \left|(\left[Z_s^k\right]^i \mid \left[Z_s^k\right]^j) - (\left[Z_s^l\right]^i \mid \left[Z_s^l\right]^j)\right| ds\right) \\ + \quad & \mathbb{E}\left(\int_0^T |f_k(B_s^y, X_s^k, Z_s^k) - f_{k,l}(B_s^y, X_s^l, Z_s^l)| ds\right). \end{aligned}$$

We know that the first expectation tends to zero; the second term tends to zero by dominated convergence (at least for a subsequence of (X^l), but it doesn't matter since (X^l) is Cauchy). Let E_1 denote the next expectation; then we can write

$$\begin{aligned} E_1 \quad & \leq \quad C\mathbb{E}\left(\int_0^T \|Z_s^l - Z_s^k\|(\|Z_s^l\| + \|Z_s^k\|) ds\right) \\ & \leq \quad \sqrt{2}C\mathbb{E}\left(\int_0^T \|Z_s^l - Z_s^k\|^2 ds\right)^{\frac{1}{2}} \mathbb{E}\left(\int_0^T (\|Z_s^l\|^2 + \|Z_s^k\|^2) ds\right)^{\frac{1}{2}}. \end{aligned}$$

The first integral tends to zero and the second is bounded because Z^l converges in L^2; hence E_1 tends to zero.

Finally, let E_2 denote the last integral; then

$$\begin{aligned} E_2 \quad \leq \quad & \mathbb{E}\left(\int_0^T |f_k(B_s^y, X_s^k, Z_s^k) - f_k(B_s^y, X_s^l, Z_s^l)| ds\right) \\ & + \mathbb{E}\left(\int_0^T |f_k(B_s^y, X_s^l, Z_s^l) - f_{k,l}(B_s^y, X_s^l, Z_s^l)| ds\right); \end{aligned}$$

now the first term goes to zero as $l \to \infty$ by dominated convergence (at least for a subsequence for which we have a.s. convergence); for the second term, the zero limit results also from dominated convergence; indeed, it suffices to write

$$\begin{aligned} & |f_k(B_s^y, X_s^l, Z_s^l) - f_{k,l}(B_s^y, X_s^l, Z_s^l)| \leq \\ & \int_{\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{ndw}} |f_k(B_s^y, X_s^l, Z_s^l) - f_k(B_s^y - \beta, X_s^l - y, Z_s^l - w)| \rho_l(\beta, y, w) d(\beta, y, w) \end{aligned}$$

and use Properties (i) and (vi) of Proposition 5.0.3.

Hence the limit in L^1 of X_t^l is

$$U - \int_t^T Z_s^k dW_s - \int_t^T \left(-\frac{1}{2}\Gamma_{ij}(X_s^k)([Z_s^k]^i \mid [Z_s]^j) + f(B_s^y, X_s^k, Z_s^k) \right) ds;$$

we know that it is also X_t^k, so by continuity :

$$a.s.,\ \forall t,\ X_t^k = U - \int_t^T Z_s^k dW_s - \int_t^T \left(-\frac{1}{2}\Gamma_{ij}(X_s^k)([Z_s^k]^i \mid [Z_s^k]^j) + f(B_s^y, X_s^k, Z_s^k) \right) ds.$$

The proof is complete.

Finally suppose that $k \to \infty$. In this case we can follow the proof above, making the changes accordingly; for instance, the expression (5.5) becomes now :

$$A_{k,k'} := \int_t^T D\hat{\Psi}(\tilde{X}_s^{k,k'}) \begin{pmatrix} (f_k - f)(V_s^k) \\ (f_{k'} - f)(V_s^{k'}) \end{pmatrix} ds;$$

remark that (5.7) is replaced by (3.2) so in particular the constant in the new inequality (5.8) does not depend on k. We give the proof of the new lemma we need :

Lemma 5.0.7 *The expectation $\mathbb{E}|A_{k,k'}|$ tends to zero as k, k' tend to infinity.*

Proof. We have

$$\begin{aligned} \mathbb{E}|A_{k,k'}| &\leq C_1 \int_0^T \mathbb{E}\left(\delta(\tilde{X}_s^{k,k'})(|f_k(V_s^k) - f(V_s^k)| + |f_{k'}(V_s^{k'}) - f(V_s^{k'})|)\right) ds; \\ &\leq C \int_0^T \mathbb{E}\Big(\delta(\tilde{X}_s^{k,k'})\big((1 + \|Z_s^k\|)(1_{\{\|Z_s^k\| \geq k\}} + 1_{\{|B_s^y| \geq k\}}) \\ &\qquad + (1 + \|Z_s^{k'}\|)(1_{\{\|Z_s^{k'}\| \geq k'\}} + 1_{\{|B_s^y| \geq k'\}})\big)\Big) ds; \end{aligned}$$

The last inequality uses the definition (5.1) of f_k and the linear growth of f (Proposition 3.3.4). As a consequence of Markov inequality and the uniform exponential integrability condition (Lemma 4.2.1), one has $1_{\{\|Z_s^k\| \geq k\}} \to 0$ when $k \to \infty$. Then the convergence to zero follows once again from dominated convergence. □

As above, this delivers a pair of processes (X, Z). To prove that this pair solves the equation $(M + D)$, we can yet follow the proof above, and the only difficulty is to deal with the term involving f. But, putting $V_s := (B_s^y, X_s, Z_s)$ and $V_s^k := (B_s^y, X_s^k, Z_s^k)$ we have

$$\begin{aligned} \mathbb{E}\left(\int_0^T |f(V_s) - f_k(V_s^k)| ds\right) &\leq \mathbb{E}\left(\int_0^T |f(V_s) - f(V_s^k)| ds\right) \\ &\quad + \mathbb{E}\left(\int_0^T |f(V_s^k) - f_k(V_s^k)| ds\right). \end{aligned}$$

Then the two terms on the right tend to zero as in the proof of Lemma 5.0.7, and this completes the proof when $k \to \infty$.

Remark : If f depends only on x (and not on z), we can take the convex function $\Psi \approx \delta^p$ of [1]. Then Hess Ψ is nonnegative and the same reasoning as above (even much simpler) gives a limit process (X_k) but not (Z_k). In fact, the only problem to deal with is to verify that X_k solves equation $(M+D)$ with drift f_k. But this results from Section 2.4 in [1], a localization argument and passing through the limit in a submartingale (using the continuity of f_k) as in the proof of Proposition 4.1.4 in [1].

6 Random terminal times

The purpose of this section is to explain how to extend the above results to the case of a random terminal time. In the Lipschitz context, this has been done in Section 5.3 in [1].

We consider here the following equation

$$(M+D)_\tau \begin{cases} dX_t = Z_t dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k|[Z_t]^j) + f(B_t^y, X_t, Z_t)\right) dt \\ X_\tau = U^\tau \end{cases}$$

where τ is a stopping time with respect to the filtration used and U^τ is a $\overline{\omega}$-valued, $\mathcal{F}_\tau$-measurable random variable. It is the counterpart of equation $(M+D)$ on the random interval $[0;\tau]$.

For bounded random terminal times (i.e. $\tau \leq T$ where T is a deterministic constant), the proofs given in the preceding sections can be used again, together with Theorem 5.3.1 of [1], which gives existence and uniqueness in the Lipschitz context. Thus we have the

Theorem 6.0.8 *We consider BSDE $(M+D)_\tau$ with $\overline{\omega} = \{\chi \leq c\}$ and $\tau \leq T$ a.s. Then under the same assumptions and in the same cases as in Theorem 3.2.1, this BSDE has a unique solution (X, Z), with $X \in \overline{\omega}$.*

Next, we consider a stopping time τ that is finite a.s. and verifies the exponential integrability condition

$$\exists \rho > 0 : \mathbb{E}(e^{\rho\tau}) < \infty. \tag{6.1}$$

Examples of such stopping times are exit times of uniformly elliptic diffusions from bounded domains in Euclidean spaces.

As in the Lipschitz case, we need to add restrictions on the drift f; indeed, the proofs above rely heavily on the construction of the submartingale $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$, so we need to keep the integrability of S_τ. Looking at the computations in the uniqueness part, the conclusion is then the same : this integrability holds for "small" drifts, i.e. there is a constant h with $0 < h < \rho$ such that, under the following condition on the constants in (3.1), (3.2) and (3.3)

$$L < h, \quad \nu > -h, \quad L_2 < h, \tag{6.2}$$

the integrability required holds, so $(S_t)_{0\leq t\leq\tau}$ is a true submartingale.

In fact we can state the following result, whose proof goes exactly the same as in Section 5.3 in [1].

Theorem 6.0.9 *We consider the BSDE $(M+D)_\tau$ with τ a stopping time verifying the integrability condition* (6.1)*; the function χ used to define the domain $\overline{\omega}$ is supposed as usual to be strictly convex. Then under the same assumptions and in the same cases as in Theorem 3.2.1, if moreover we suppose that f is "small" (i.e. verifies condition* (6.2) *above), this BSDE has a unique solution (X, Z).*

Remark : Of course the applications to PDEs in [1] are still valid under the current assumptions.

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